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1 Literature and Links

- The Simple Theory of Types: [Church, 1940]
- \mathcal{Q}_0 : [Andrews, 2002, pp. 210–215; first edition Andrews, 1986, pp. 161–165; very first publication in Andrews, 1963, pp. 345 f., 350]. Elementary logic is developed without any appeal to the Axiom of Choice, which may be assumed later [Andrews, 2002, p. 236]. Adding an Axiom of Infinity (as Axiom 6) for number theory yields \mathcal{Q}_0^∞ [Andrews, 2002, p. 259]. A short description [Andrews, 2014a] is available online at:
<http://plato.stanford.edu/entries/type-theory-church/#ForBasEqu>
- HOL: [Gordon, 2001] (revised version of the 1985 paper), [Gordon and Melham, 1993].¹ The part on the logic written by Andrew Pitts² is available online (with minor modifications) at:
<http://freifr.sourceforge.net/project/hol/hol/kananaskis-10/kananaskis-10-logic.pdf>
 For a historical account on the HOL family, see [Gordon, 2000].
- HOL4: [Slind and Norrish, 2008], <http://hol-theorem-prover.org>
- HOL Light: [Harrison, 2009],³ <http://www.cl.cam.ac.uk/~jrh13/hol-light/>
- HOL Zero: [Adams, 2010], <http://www.proof-technologies.com/holzero/>
- Isabelle/HOL: [Paulson, 1989], <http://www.cl.cam.ac.uk/research/hvg/Isabelle/>
- ProofPower: [Arthan and Jones, 2005], <http://www.lemma-one.com/ProofPower/index/>
- \mathcal{Q} : [Andrews, 1965]
- Extended HOL: [Melham, 1993b]
- F_ω : [Girard, 1986]⁴

¹Further definitional principles are proposed and discussed in [Kumar et al., 2014], [Arthan, 2014], and [Arthan, 2016].

²Part III: The HOL Logic [Gordon and Melham, 1993, pp. 191–232]. “Part III presents the logic supported by HOL (‘The HOL Logic’). It consists of two chapters: an informal introduction and a formal set-theoretic semantics (written by Andrew Pitts).” [Gordon and Melham, 1993, p. xv] In an email to the author, Andrew Pitts confirmed that originally he wrote the material in the file available online, which contains both (!) chapters.

³Further motivation for HOL Light is explained in [Harrison, 1995].

⁴For an introduction, see also chapter 30 of [Pierce, 2002, pp. 449 ff.].

- HOL2P: [Völker, 2007], <http://cswww.essex.ac.uk/staff/norbert/hol2p/>
- HOL-Omega (HOL_ω): [Homeier, 2009], <http://www.trustworthytools.com/id17.html>
- Automath: [Daalen, 1994],⁵ <http://www.win.tue.nl/automath/>
A modern reimplementation: [Wiedijk, 2002], <http://www.cs.ru.nl/~freek/aut/>
- PVS: [Owre, Rushby, and Shankar, 1992], <http://pvs.csl.sri.com>
- \mathcal{R}_0 : [Kubota, 2015], <http://doi.org/10.4444/100.10>
- Coq: [Coquand and Huet, 1985], [Coquand and Huet, 1988], [Coquand and Paulin, 1990], <http://coq.inria.fr>
- Nuprl: [Constable, 1986], <http://www.nuprl.org>
- Agda: [Norell, 2007],⁶ <http://wiki.portal.chalmers.se/agda/>⁷

Please note that logics and their implementation below the expressiveness of higher-order logic (e.g., first-order logic) are not considered.

Neither the graph nor the above list are comprehensive. Further logics are William M. Farmer’s \mathcal{Q}_0^u and \mathcal{Q}_0^{uqe} (extensions of Andrews’ \mathcal{Q}_0 with undefinedness). Further type-theoretic systems are TPS by Peter B. Andrews et al., IMPS (with partial functions and subtypes) by William M. Farmer et al., and *Prooftoys* (a natural deduction variant of Andrews’ \mathcal{Q}_0) by Cris Perdue, and further set-theoretic systems are *Mizar* by Andrzej Trybulec and *Metamath* by Norman Megill. Historically, “[t]he original LCF team at Stanford consisted of Robin Milner, assisted by Whitfield Diffie” [Gordon, 2000, p. 169] – now known for the Diffie-Hellman (DH) key exchange in cryptography. Both Rob Arthan and Roger Bishop Jones generously credit each other for having the main rôle in the creation of ProofPower.

- \mathcal{Q}_0^u : [Farmer, 2014]
- \mathcal{Q}_0^{uqe} : [Farmer, 2015]
- TPS: <http://gtps.math.cmu.edu/tps.html>
- IMPS: <http://imps.mcmaster.ca>
- Prooftoys (and Mathtoys): <http://prooftoys.org>, <http://mathtoys.org>
- Mizar: <http://mizar.org>
- Metamath: <http://metamath.org>

⁵“The standard reference about Automath is [Nederpelt, Geuvers, and Vrijer, 1994]. This is a compilation of almost all important Automath publications. It contains as (A.3) a good introduction to Automath [Daalen, 1994]. The main important Automath paper that is missing from this collection is the paper [de Bruijn, 1991] about telescopes.” [Wiedijk, 2002, p. 365]

⁶“The particular choice of type theory is not crucial and the theory we choose is roughly Luo’s UTT [Luo, 1994] extended with Σ -types and η -laws.” [Norell, 2007, p. 14]

⁷The current version is Agda 2. For Agda version 1, see: <http://ocvs.cfv.jp/Agda/>.

A more comprehensive list with descriptions is available at: <http://www.nuprl.org/Intro/others.html>.

For a discussion, and a comparison with set-theoretic approaches such as Mizar, see [Wiedijk, 2007].

For a survey on interactive theorem proving, see [Harrison, Urban, and Wiedijk, 2014], which includes a description of the recently completed formalization of the proof of the *Kepler conjecture*. Despite the currently limited expressiveness of the formal language (lacking explicit quantification over type variables and dependent types⁸), in 2014, theorem provers verified this proof at the front of mathematical research established by Thomas Hales together with Ferguson in 1998 and published in full in 2006.⁹

2 Design Decisions

- Alonzo Church on the *reduction to functions of one argument*: “Adopting a device due to Schönfinkel, we treat a function of two variables as a function of one variable whose values are functions of one variable, and a function of three or more variables similarly.” [Church, 1932, p. 352]
- Alonzo Church *introducing λ -notation*: “If \mathbf{M} is any formula containing the variable \mathbf{x} , then $\lambda\mathbf{x}[\mathbf{M}]$ is a symbol for the function whose values are those given by the formula.” [Church, 1932, p. 352]
- Peter B. Andrews on the *description operator* (and Axiom 5 of \mathcal{Q}_0): “Henkin remarks at the end of [Henkin, 1963] that when one passes from the theory of propositional types to the full theory of finite types, it becomes necessary to add a constant $\iota_{1(01)}$ to denote a descriptor function, and an appropriate axiom involving this constant. We note that for this axiom it suffices to take the simple formula $\iota_{1(01)}(\lambda x_1(x_1 \equiv y_1)) \equiv y_1$ [...]” [Andrews, 1963, p. 350]
- Peter B. Andrews on *finding as simple, natural, and expressive a formulation of type theory as possible*: “Chapter 5 introduces a system \mathcal{Q}_0 of typed λ -calculus which is essentially the system introduced by Alonzo Church in [Church, 1940], except that [...] equality relations rather than quantifiers and propositional connectives are primitive. [...] The discussion turns to finding as simple, natural, and expressive a formulation of type theory as possible. [...] This leads to an exposition of the basic ideas underlying \mathcal{Q}_0 .” [Andrews, 2002, p. xvi]
- Peter B. Andrews on *expressiveness* as the main criterion for the design of \mathcal{Q}_0 : “Therefore we shall turn our attention to finding a formulation of type theory which is as expressive as possible, allowing mathematical ideas to be expressed precisely with a minimum of circumlocutions, and which is as simple and economical as is possible without sacrificing expressiveness. The reader will observe that the formal language we find arises very naturally from a few fundamental design decisions.” [Andrews, 2002, p. 205 f.]
- Peter B. Andrews on *equality as the basic primitive notion*: “A formulation of type theory based on these ideas was introduced by Church in [Church, 1940], and proved complete by Henkin in

⁸“The theorem prover HOL Light [...] is based on a logic without dependent types, but we can still encode the index N as a type (roughly, an arbitrary indexing type of size N).” [Hales and Harrison, 2010, p. 6]

⁹The computer aided verification became a desire, as the traditional peer review of the proof of about 300 pages and 40,000 lines of additional computer code had to be given up after four years of team efforts. Initiated by Hales in 2006 under the project name *Flyspeck*, the verification was completed successfully in 2014, proving the *Kepler conjecture* from 1611 (forming a part of Hilbert’s 18th problem) which “states that no arrangement of identical balls in ordinary 3-dimensional space has a higher packing density than the obvious cannonball arrangement” [Harrison, Urban, and Wiedijk, 2014, p. 192] with a density of $\pi/\sqrt{18}$ ($\approx 74\%$). “The formal verifications have been carried out in the HOL Light and Isabelle proof assistants, with a second verification of the main statement in HOL Zero.” [Hales, 2015, p. 2]

[Henkin, 1950]. In this system equality can be defined using connectives and quantifiers. However, it is also possible to define connectives and quantifiers in terms of equality. Equality is a very basic and simple notion, so instead of using Church’s formulation of type theory, we shall use a slight variant of it (first introduced in [Henkin, 1963] and simplified in [Andrews, 1963]) in which equality is taken as the basic primitive notion.” [Andrews, 2002, p. 208]

- Peter B. Andrews introducing *ordered pairs (without extending the formal language)*: “The expression $[\lambda g_{ooo} \cdot g_{ooo} x_o y_o]$ can be used to represent the ordered pair $\langle x_{[o]}, y_o \rangle$ ”. [Andrews, 2002, p. 208]
- Peter B. Andrews introducing *induction* as part of the definition of natural numbers (*without extending the formal language*): “The Induction Principle simply limits the size of the set.” [Andrews, 2002, p. 259] “We next define the natural numbers. These are equivalence classes of sets of individuals, and so have type $(o(oI))$.”

DEFINITIONS. Let σ be the type symbol $(o(oI))$.

[...]

$\mathbb{N}_{o\sigma}$ stands for $[\lambda n_\sigma \forall p_{o\sigma} \cdot [p_{o\sigma} 0_\sigma \wedge \forall x_\sigma \cdot p_{o\sigma} x_\sigma \supset p_{o\sigma} S_{\sigma\sigma} x_\sigma] \supset p_{o\sigma} n_\sigma]$

[...]

[...] The wff S represents the successor function. [...] $\mathbb{N}_{o\sigma}$ represents the set of natural numbers, i.e., the intersection of all sets which contain 0 and are closed under S .” [Andrews, 2002, p. 260]

- Peter B. Andrews introducing *recursion* by defining a recursion operator (*without extending the formal language*): “Indeed, since f is uniquely determined by h and g , we can define a recursion operator R such that $f = Rhg$.” [Andrews, 2002, p. 281]

“DEFINITION. $R_{\sigma\sigma\sigma(\sigma\sigma\sigma)}$ stands for $[\lambda h_{\sigma\sigma\sigma} \lambda g_\sigma \lambda n_\sigma \cdot \lambda m_\sigma \forall w_{o\sigma\sigma} \cdot [w_{o\sigma\sigma} 0_\sigma g_\sigma \wedge \forall x_\sigma \forall y_\sigma \cdot w_{o\sigma\sigma} x_\sigma y_\sigma \supset w_{o\sigma\sigma} [S_{\sigma\sigma} x_\sigma] h_{\sigma\sigma\sigma} x_\sigma y_\sigma] \supset w_{o\sigma\sigma} n_\sigma m_\sigma]$.”

6400. $\vdash [R_{\sigma\sigma\sigma(\sigma\sigma\sigma)} h_{\sigma\sigma\sigma} g_\sigma] 0_\sigma = g_\sigma \wedge \forall n_\sigma \cdot [Rh_{\sigma\sigma\sigma} g_\sigma][S_{\sigma\sigma} n] = h_{\sigma\sigma\sigma} n_\sigma \cdot [Rh_{\sigma\sigma\sigma} g_\sigma] n_\sigma$ ”

[Andrews, 2002, p. 282, boldface as in the original] (“We [...] let $[\gamma \mathbf{z}_\gamma \mathbf{A}]$ stand for $\iota_{\gamma(o\gamma)}[\lambda \mathbf{z}_\gamma \mathbf{A}]$.”

[Andrews, 2002, p. 234, boldface as in the original]. For the definition of the universal quantifier on numbers \forall , see [Andrews, 2002, p. 260].)

“To illustrate the usefulness of R , define $+_{\sigma\sigma\sigma}$ as $R_{\sigma\sigma\sigma(\sigma\sigma\sigma)}[\lambda x_\sigma S_{\sigma\sigma}]$, and $\mathbf{A}_\sigma + \mathbf{B}_\sigma$ as $+_{\sigma\sigma\sigma} \mathbf{A}_\sigma \mathbf{B}_\sigma$. [...] Similarly, $\times_{\sigma\sigma\sigma}$ can be defined as $[\lambda m_\sigma \cdot R_{\sigma\sigma\sigma(\sigma\sigma\sigma)}[\lambda x_\sigma \cdot +_{\sigma\sigma\sigma} m_\sigma] 0_\sigma]$, and $\mathbf{A}_\sigma \times \mathbf{B}_\sigma$ as $\times_{\sigma\sigma\sigma} \mathbf{A}_\sigma \mathbf{B}_\sigma$.” [Andrews, 2002, p. 284, boldface as in the original]

- Andrew M. Pitts on the *natural deduction* system HOL in comparison to the *axiomatic (Hilbert-style) deductive system* \mathcal{Q}_0 : “From a logical point of view, it would be better to have a simpler substitution primitive, such as ‘Rule R’ of Andrews’ logic \mathcal{Q}_0 , and then to derive more complex rules from it.” [Gordon and Melham, 1993, p. 213]

- Mike Gordon characterizing the *HOL logic* (without mentioning the definability of new types): “The HOL logic is a version of Church’s simple theory of types [Church, 1940] modified by:
 - allowing types to contain variables (i.e. be *polymorphic*) and
 - simplifying the axioms and rules of inference.” [Gordon, 2001, p. 4]

- Mike Gordon and Peter B. Andrews defining the conditional using *the epsilon operator vs. the description operator*.

Mike Gordon (HOL): “Many things that are normally primitive can be defined using the ϵ -operator. For example, the conditional term $\text{Cond } t \ t_1 \ t_2$ (meaning ‘if t then t_1 else t_2 ’) can be defined by:

$$\text{Cond } t \ t_1 \ t_2 = \epsilon x.((t = \mathbf{T}) \Rightarrow (x = t_1)) \wedge ((t = \mathbf{F}) \Rightarrow (x = t_2))” [Gordon, 2001, p. 24]$$

Peter B. Andrews (\mathcal{Q}_0): “DEFINITION: Let $C_{\gamma o\gamma\gamma}$ be

$$[\lambda x_\gamma \lambda y_\gamma \lambda p_o \cdot \lambda q_\gamma \cdot [p_o \wedge \cdot x_\gamma = q_\gamma] \vee [\sim p_o \wedge \cdot y_\gamma = q_\gamma]]$$

$C_{\gamma o \gamma \gamma} x_{\gamma} y_{\gamma} p_o$ can be read ‘if p_o then x_{γ} , else y_{γ} ’.

5313. $\vdash [C_{\gamma o \gamma \gamma} x_{\gamma} y_{\gamma} T_o = x_{\gamma}] \wedge \blacksquare C_{\gamma o \gamma \gamma} x_{\gamma} y_{\gamma} F_o = y_{\gamma}$ [Andrews, 2002, p. 235, boldface as in the original]

Formal verification of theorem **5313** in the \mathcal{R}_0 implementation (with type abstraction):

“ $:= CHOOS \dots$

$\dots [\lambda t_{\tau}. [\lambda x_t. [\lambda y_t. [\lambda p_o. (\iota_{t(o)} [\lambda q_t. (\vee_{ooo} (\wedge_{ooo} p_o (=_{ott} x_t q_t))) (\wedge_{ooo} (\sim_{oo} p_o) (=_{ott} y_t q_t))])_o] t]_{(to)}]_{(tot)}]_{(tott)}$
 $\# \text{ wff } 2080 : [\lambda t. [\lambda x. [\lambda y. [\lambda p. (\iota [\lambda q. (\vee (\wedge p (= x q)) (\wedge (\sim p) (= y q)))]))]]]]_{\setminus 4o \setminus 3 \setminus 2\tau} [\dots]$
 $[\dots]$

$\# \wedge (= (CHOOS t x y T) x) (= (CHOOS t x y F) y) := A5313$

$\# \wedge_{ooo} (=_{ott} (CHOOS_{\setminus 4o \setminus 3 \setminus 2\tau} t x y T_o) x_t) (=_{ott} (CHOOS_{\setminus 4o \setminus 3 \setminus 2\tau} t x y F_o) y_t)$

$:= A5313$ ” [Kubota, 2016, pp. 165, 174]

- Mike Gordon on *explicit type variable quantification* in HOL: “In future versions of HOL it is expected that there will be explicit type variable quantification [Melham, 1993b], i.e., terms of the form $\forall \alpha.t$ (where α is a type variable). The right hand side of definitions will be required to be closed with respect to both term and type variables. Melham has shown that this will make defining mechanisms much cleaner and also permit an elegant treatment of type specifications.” [Gordon, 2000, p. 175, fn. 7]
- Tom Melham on *quantification over type variables at the level of terms (not types)*: “Note that we are not proposing an extension to the type language of HOL – the quantifications $\forall \alpha.P$ and $\exists \alpha.P$ are new term constructs, and not type constructs of the kind found (for example) in Girard’s system F [...]. The extended logic proposed here resembles system \mathcal{Q} , a transfinite type theory due to Andrews [Andrews, 1965]. It is, however, still much weaker than Andrews’ system.” [Melham, 1993b, p. 8 f., emphasis as in the original]
- Tom Melham on *the relation of Melham’s Extended HOL to Andrews’ system Q*: “I have found Andrews’ book [Andrews, 1965] invaluable in working out many of the technical details of the extension to HOL proposed here.” [Melham, 1993b, p. 23]
- Mike Gordon suggesting *type abstraction* for HOL: “The syntax and semantics of type variables are currently being studied by several logicians. A closely related area is the theory of ‘second order’ λ -terms like $\lambda \alpha. \lambda x:\alpha. x$, perhaps such terms should be included in the HOL logic.” [Gordon, 2001, p. 22]
- John Harrison on the *motivation for HOL Light*: “Part of my objective in HOL Light was to arrive at a simpler and more satisfying logical kernel that could nevertheless be the foundation of a practical tool. In particular HOL Light’s foundations address a couple of the issues [mentioned in] some self-criticism [in] the HOL documentation: the complicated primitive substitution rule and [...] the way the epsilon operator is plumbed deep into the foundations. [...]
 [...]
 In HOL Light the basic logical notions including the quantifiers are all defined independent of the epsilon operator (in contrast to its use even to define ‘there exists’ in the original HOL). The choice operator is eventually introduced but it is quite late. [...]
 [...]
 Ultimately, all the quantifiers and connectives in HOL Light are defined in terms of equality alone. In this respect the foundations are reminiscent of one of Andrews’[...] systems, but with the distinction that the definitions are intuitionistically admissible. [...]
 [...]
 I think there is something quite satisfying about the derivation of higher-order logic from these simple foundations based just on equality in a simple world of functions. [...]” [Harrison, 2016]

- Larry Paulson on the relevance of *natural deduction* for *automation*: “Church’s axiom system is now antiquated, largely dating back to *Principia Mathematica*. There are improved formulations but most use the Hilbert style. Natural deduction is far superior for automated proof.” [Paulson, 1989, p. 3, emphasis as in the original]
- Sam Owre and Natarajan Shankar on *(non-structurally) dependent types* in PVS: “One very important consequence of the above extension of the universe is that all type dependencies must be bounded [...]. This property is easily proved by induction on the structure of a PVS type [...]. In particular, there is no way to define a type constructor T^n in PVS [...]. If unbounded type dependencies were allowed in PVS, one can construct a dependent type such as $[n : nat \rightarrow T^n]$ whose representation is not in U as defined above.” [Owre and Shankar, 1999, p. 35, emphasis as in the original] “PVS admits only a very restricted form of type dependency. In a dependent type $T(n)$, the parameter n can occur only within subtype predicates in $T(n)$. This means that the structure of $T(n)$ is invariant with respect to n . All possible ways of introducing type dependencies in PVS preserve this invariant. It follows that there is no way of defining a type $T(n)$, where $T(n)$ is A^n , i.e., the n -tuple over the type A .” [Shankar and Owre, 2000, p. 45, emphasis as in the original]
- Robert L. Constable et al. on choosing a *predicative type structure* for Nuprl: “The type structure hierarchy of Nuprl resembles that of *Principia Mathematica*, the ancestor of all type theories. The hierarchy manifests itself in an unbounded cumulative hierarchy of universes, U_1, U_2, \dots , where by cumulative hierarchy we mean that U_i is in U_{i+1} and that every element of U_i is also an element of U_{i+1} . Universes are themselves types, and every type occurs in a universe. In fact, A is a type if and only if it belongs to a universe. Conversely all the elements of a universe are types.” [Constable, 1986, p. 5, emphasis as in the original] “The concept of a universe in this role, to organize the hierarchy of types, is suggested in [Artin, Grothendieck, & Verdier 72] and was used by Martin-Löf [Martin-Löf 73]. This is a means of achieving a *predicative* type structure as opposed to an *impredicative* one as in Girard [Girard 71] or Reynolds [Reynolds 83].” [Constable, 1986, p. 5, fn. 2, emphasis as in the original]
- Gérard Huet on the *development of COQ*: “The logical language used by COQ is a variety of type theory, called the *Calculus of Inductive Constructions*. [...] The λ -calculus notation, originally used for expressing functionality, could also be used as an encoding of natural deduction proofs. This Curry-Howard isomorphism was used by N. de Bruijn in the *Automath* project, the first full-scale attempt to develop and mechanically verify mathematical proofs. [...] Exploiting this Curry-Howard isomorphism, notable achievements in proof theory saw the emergence of two type-theoretic frameworks; the first one, Martin-Löf’s *Intuitionistic Theory of Types*, attempts a new foundation of mathematics on constructive principles. The second one, Girard’s polymorphic λ -calculus F_ω , is a very strong functional system in which we may represent higher-order logic proof structures. Combining both systems in a higher-order extension of the Automath languages, T. Coquand presented in 1985 the first version of the *Calculus of Constructions*, CoC. [...] The formalism was extended in 1989 by T. Coquand and C. Paulin with primitive inductive definitions, leading to the current *Calculus of Inductive Constructions*. [...] [...] A first implementation of CoC was started in 1984 by G. Huet and T. Coquand. [...]” [Huet, 1995, emphasis as in the original]
- Diederik van Daalen on *coding* logic using the *propositions-as-types* notion: “Now there are different ways of coding some logic into the objects-and-types framework. Here we only mention a so-called *functional interpretation* of logic, which gives rise to the *propositions-as-types* notion.

This idea of interpreting logic was developed independently by de Bruijn and certain others, of whom we mention Howard [Howard, 1980], Prawitz [Prawitz 71], Girard [Girard 72] and Martin-Löf [Martin-Löf 75a].” [Daalen, 1994, p. 111, emphasis as in the original]

- Ana Bove and Peter Dybjer on *variants of Martin-Löf’s intuitionistic type theory*: “The Curry-Howard interpretation was the basis for Martin-Löf’s intuitionistic type theory [19,20,21]. In this theory propositions and types are actually identified. Although Martin-Löf’s theory was primarily intended to be a foundational system for constructive mathematics, it can also be used as a programming language [20]. From the early 1980’s and onwards, a number of computer systems implementing variants of Martin-Löf type theory were built. The most well-known are the NuPRL [Constable, 1986] system from Cornell implementing an extensional version of the theory, and the Coq [32] system from INRIA in France implementing an intensional impredicative version. The Agda system implements an intensional predicative extension of Martin-Löf type theory. It is the latest in a sequence of systems developed in Göteborg.” [Bove and Dybjer, 2009, p. 59]
- Ulf Norell about using *encoding properties of values as types*: “Since dependent types allows types to talk about values, we can encode properties of values as types whose elements are proofs that the property is true. This means that a dependently typed programming language can be used as a logic.” [Norell, 2009, p. 230]
- Mark Adams introducing the notion of *faithfulness*: “Also note that none of these notions fully address the issue of *faithfulness*, where internal representation and concrete syntax correctly correspond. A printer that printed `false` as `true` and `true` as `false` might be Pollack-consistent but would not be faithful.” [Adams, 2016, p. 21, emphasis as in the original]
- Mark Adams on the *Pollack-consistency* of HOL Zero (not taking into account unpublished \mathcal{R}_0): “We believe HOL Zero does not suffer from any incompleteness or ambiguity in its parsers or printers, and printed output can always be parsed back in to give the same internal representation. This would make HOL Zero’s parsers and printers well-behaved and Pollack-consistent. As far as we know, this would be a first amongst not only HOL systems, but also various other theorem proving systems that support concrete syntax, such as COQ and Mizar.” [Adams, 2016, p. 34]

3 Criticism

- Mike Gordon on the use of the *epsilon operator* in HOL: “It must be admitted that the ε -operator looks rather suspicious.” [Gordon, 2001, p. 24] “The inclusion of ε -terms into HOL ‘builds in’ the Axiom of Choice [...]” [Gordon, 2001, p. 24]
- Freek Wiedijk on the *(current) HOL family*: “There is one important reason why the HOLs are not yet attractive enough to be taken to be the QED system:

- The HOL type system is too poor. As we already argued in the previous section, it is too weak to properly do abstract algebra.

But it is worse than that. In the HOL type system there are no *dependent types*, nor is there any form of *subtyping*. (Mizar and Coq both have dependent types and some form of subtyping. In Mizar the subtyping is built into the type system. In Coq a similar effect is accomplished through the use of automatic *coercions*.)

For formal mathematics it is essential to both have dependent types and some form of subtyping.” [Wiedijk, 2007, p. 130, emphasis as in the original]

- Freek Wiedijk on the (*current*) *Coq* system: “There are two important reasons why Coq is not yet attractive enough to be taken to be the QED system:
 - The foundations of Coq are too complicated. They are baroque to say the least. Maybe they are even *beyond* baroque. They might even be called *rococo*.
There is no paper in which the foundations of Coq are spelled out in full mathematical precision. [...]
[...]
Also, the foundations of Coq are sufficiently complicated that they are tinkered with, and therefore change between versions of the system.
 - As already noted in the previous section, Coq is not designed for classical mathematics, which means that doing classical & extensional mathematics in it is not as easy as one would like it to be.” [Wiedijk, 2007, p. 130, emphasis as in the original]
- John Harrison, Josef Urban, and Freek Wiedijk on the (*current*) *HOL* family: “Another limitation of the simple HOL type system is that there is no explicit quantifier over polymorphic type variables, which can make many standard results like completeness theorems and universal properties awkward to express, though there are extensions with varying degrees of generality that fix this issue [Melham, 1993a; Völker, 2007; Homeier, 2009]. Inflexibilities of these kinds certainly arise in simple type theories, and it is not even clear that more flexible *dependent* type theories (where types can be parametrized by terms) are immune. For example, in one of the most impressive formalization efforts to date [Gonthier *et al.*, 2013] the entire group theory framework is developed in terms of subsets of a single universe group, apparently to avoid the complications from groups with general and possibly heterogeneous types.” [Harrison, Urban, and Wiedijk, 2014, pp. 170 f., emphasis as in the original]
- John Harrison, Josef Urban, and Freek Wiedijk on the (*current*) *Coq* system: “Indeed, there does not seem to be any precise written specification of Coq’s current foundations, other than the actual code.” [Harrison, Urban, and Wiedijk, 2014, p. 188, fn. 29]
- Peter B. Andrews on a *false soundness assertion in Henkin’s article on completeness*: “As shown in [Andrews, 1972b], there is a nonstandard general model for \mathcal{C} in which the Axiom of Extensionality $\forall x_\beta (f_{\alpha\beta}x_\beta = g_{\alpha\beta}x_\beta) \supset (f_{\alpha\beta} = g_{\alpha\beta})$ is not valid, since the sets in this model are so sparse that the denotation of the defined equality formula $\mathcal{L}_{\alpha\alpha}$ is not the actual equality relation. Thus, Theorem 2 of [Henkin, 1950] (which asserts the completeness and soundness of \mathcal{C}) is technically incorrect. The apparently trivial soundness assertion is false.” [Andrews, 2014b, p. 70] (Note that [Andrews, 1972b] bases on the preceding [Andrews, 1972a].)

4 Historical Notes

- Alonzo Church *first proposing a third alternative to Russell’s type theory and Zermelo’s axiomatic set theory* in 1932: “Rather than adopt the method of Russell for avoiding the familiar paradoxes of mathematical logic, or that of Zermelo, both of which appear somewhat artificial, we introduce for this purpose, as we have said, a certain restriction on the law of excluded middle.” [Church, 1932, p. 347]
- Felice Cardone and J. Roger Hindley on the *invention of λ -calculus and the origin of the λ -notation*: “The λ -calculus was invented in about 1928 by Alonzo Church, and was first

published in [Church, 1932]. Church was born in 1903 in Washington D.C. and studied at Princeton University. He made his career at Princeton until 1967, though in 1928–29 he visited Göttingen and Amsterdam.

Around 1928 he began to build a formal system with the aim of providing a foundation for logic which would be more natural than Russell’s type theory or Zermelo’s set theory, and would not contain free variables (for reasons he explained in [Church, 1932, pp. 346–347]). He chose to base it on the concept of function rather than set, and his primitives included abstraction $\lambda x[M]$ and application $\{F\}(X)$, which we shall call here ‘ $\lambda x.M$ ’ and ‘ (FX) ’.

(By the way, why did Church choose the notation ‘ λ ’? In [Church, 1964, §2] he stated clearly that it came from the notation ‘ \hat{x} ’ used for class-abstraction by Whitehead and Russell, by first modifying ‘ \hat{x} ’ to ‘ $\wedge x$ ’ to distinguish function-abstraction from class-abstraction, and then changing ‘ \wedge ’ to ‘ λ ’ for ease of printing. This origin was also reported in [Rosser, 1984, p.338]. On the other hand, in his later years Church told two enquirers that the choice was more accidental: a symbol was needed and ‘ λ ’ just happened to be chosen.)

As mentioned earlier, Church was not the first to introduce an explicit notation for function-abstraction. But he was the first to state explicit formal conversion rules for the notation, and to analyse their consequences in depth.” [Cardone and Hindley, 2009, pp. 730 f.]

- Felice Cardone and J. Roger Hindley on the *evolution of Church’s simple type theory*: “Church’s simple type theory was a function-based system, stemming from ideas of Frank Ramsey and Leon Chwistek in the 1920s, for simplifying the type theory of [Russell and Whitehead, 1913]. Church lectured on his system in Princeton in 1937–38 before publishing it in [Church, 1940], and his lectures were attended by Turing, who later made some technical contributions. [...] Church’s system was analysed and extended in a series of Princeton Ph.D. theses from the 1940s onward, of which perhaps the best known are Leon Henkin’s in 1947, published in [Henkin, 1950], and Peter Andrews’, published in [Andrews, 1965]. Henkin gave two definitions of model of typed λ (*standard* and *general* models), and proved the completeness of simple type theory with respect to general models. Andrews extended Church’s system to make a smooth theory of tran[s]finite types.” [Cardone and Hindley, 2009, p. 737, emphasis as in the original]
- Peter B. Andrews on *Henkin’s rôle for the development of \mathcal{Q}_0* : “Quine described how to make these definitions in the short final section of [16], but Henkin developed this topic much further in [Henkin, 1963], introducing an axiomatic system and establishing its soundness and completeness. [...] [...] Henkin’s work played a decisive role in my life. [...] Henkin’s work developing a formulation of Church’s type theory with equality (identity) as the sole logical primitive was particularly important for me. I used such a formulation of full type theory, called \mathcal{Q}_0 , in my Ph.D. thesis [Andrews, 1965] and the textbook [Andrews, 2002]. [...] It is important to realize the significance of Henkin’s contribution in developing a formulation of type theory based on equality. It is a real improvement of the system \mathcal{C} discussed in [Henkin, 1950], which has primitive constants for propositional connectives and quantifiers [...]” [Andrews, 2014b, pp. 68 f.]
- Andrew M. Pitts on *type variables in HOL*: “In Church’s original formulation of simple type theory, type variables are part of the meta-language and are used to range over object language types. Proofs that contain type variables were understood as proof schemes (i.e. families of proofs). To support such proof schemes *within* the HOL logic, type variables have been added to the object language type system.” [Gordon and Melham, 1993, p. 195, emphasis as in the

original]

- Andrew M. Pitts on the *set of rules and axioms* for HOL: “The particular set of rules and axioms chosen to axiomatize the HOL logic is rather arbitrary. It is partly based on the rules that were used in the LCF logic $PP\lambda$, since HOL was implemented by modifying the LCF system. In particular, the substitution rule `SUBST` is exactly the same as the corresponding rule in LCF; the code implementing this was written by Robin Milner and is highly optimized. Because substitution is such a pervasive activity in proof, it was felt to be important that the system primitive be as fast as possible. From a logical point of view, it would be better to have a simpler substitution primitive, such as ‘Rule R’ of Andrews’ logic Q_0 , and then to derive more complex rules from it.” [Gordon and Melham, 1993, p. 213]
- Mike Gordon on the *genesis of HOL*: “[...] [T]he terms [...] could be encoded [...] in such a way that the LSM expansion-law just becomes a derived rule [...]. This approach is both more elegant and rests on a firmer logical foundation, so I switched to it and HOL was born. [...] The logic supported by Cambridge LCF has the usual formula structure of predicate calculus, and the term structure of the typed λ -calculus. The type system, due to Milner, is essentially Church’s original one [Church, 1940], but with type variables moved from the meta-language to the object language (in Church’s system, a term with type variables is actually a meta-notation – a term-schema – denoting a family of terms, whereas in LCF it is a single polymorphic term). [...] [...] HOL employs the LCF substitution because I wanted to use the existing efficient code. As a result the HOL logic ended up with a rather *ad hoc* formal basis. Another inheritance from LCF is the use of a natural deduction logic (Church used a Hilbert-style formal system). [...]” [Gordon, 2000, p. 173, emphasis as in the original]
- Mike Gordon on the *development of HOL*: “The design of HOL was largely taken ‘off the shelf,’ the theory being classical higher order logic and the implementation being LCF. The development of the system was, at first, primarily driven by hardware verification case studies.” [Gordon, 2000, p. 174]
- Mike Gordon on the *validation of HOL’s definitional principles*: “[...] Dr Andrew Pitts was commissioned to validate HOL’s definitional principles. He produced informal proofs that they could not introduce inconsistency [Gordon and Melham, 1993, Chapter 16].” [Gordon, 2000, p. 175]
- Ondřej Kunčar and Andrei Popescu *classifying the HOL logic*: “Polymorphic HOL, more precisely, Classic[al] Higher-Order Logic with Infinity, Hilbert Choice and Rank-1 Polymorphism, endowed with a mechanism for constant and type definitions, was proposed in the [eigh]ties as a logic for interactive theorem provers by Mike Gordon, who also implemented the seminal HOL theorem prover [Gordon and Melham, 1993].” [Kunčar and Popescu, 2015, p. 234]

For a detailed treatment of epsilon terms, see [Slater, 2016].

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